ASTR 421

Stellar Observations and Theory

Lecture 13
Stellar Structure: II

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Today

- Stellar Structure Equations
- Polytropes
- The Lane-Emden Equation

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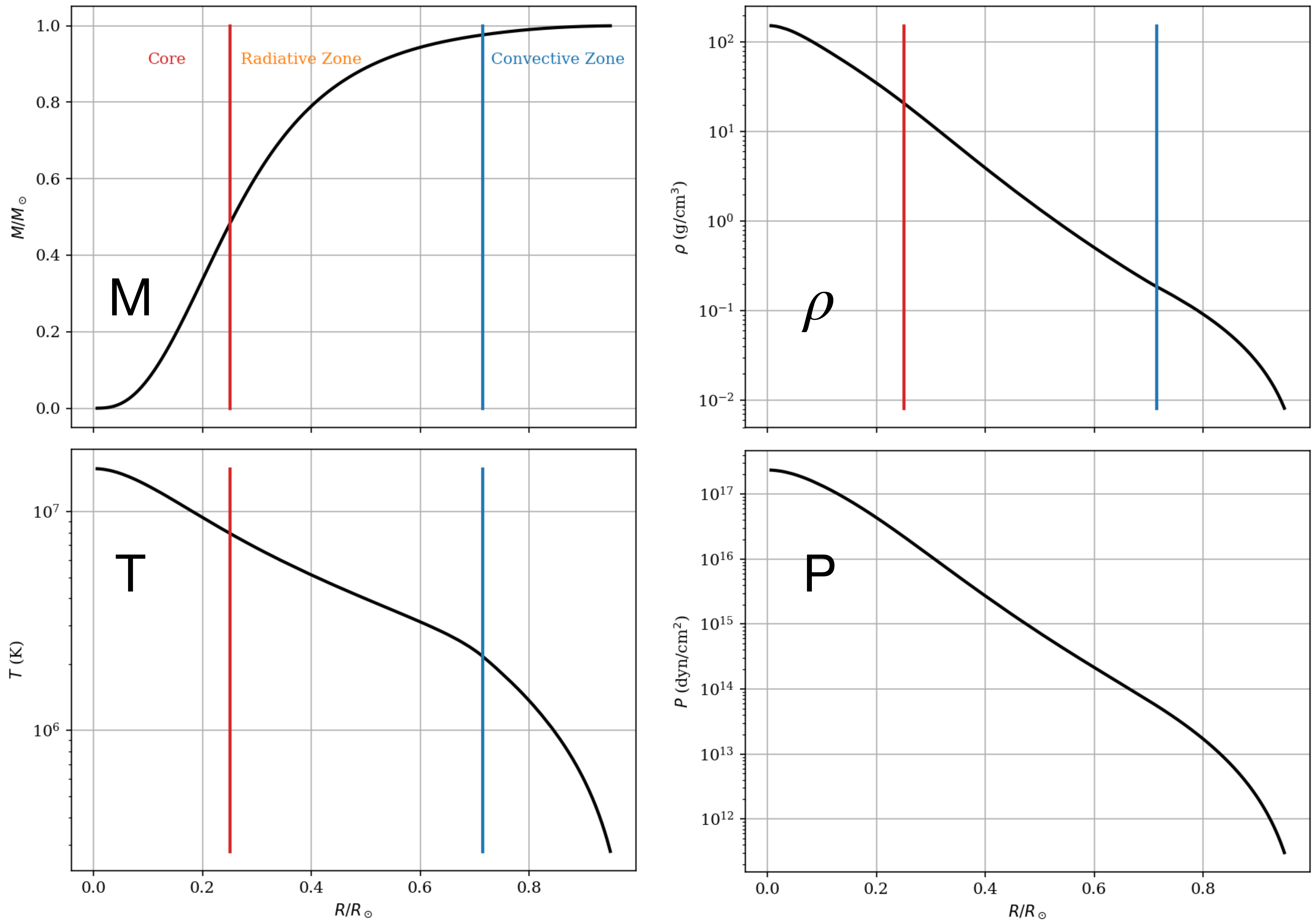
Stellar Structure Equations



- Last time we looked at *solar* structure qualitatively

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Stellar Structure Equations

- The goal is to write equations that describe the state of gas inside a star, making some simplifying assumptions, including:
  - Spherical symmetry
  - Steady state
  - LTE
- Even “simple” models can be very complex, esp. when we start adding effect of opacity, composition
- Let’s look at the general equations first, and then a particular set of solutions (polytropes)

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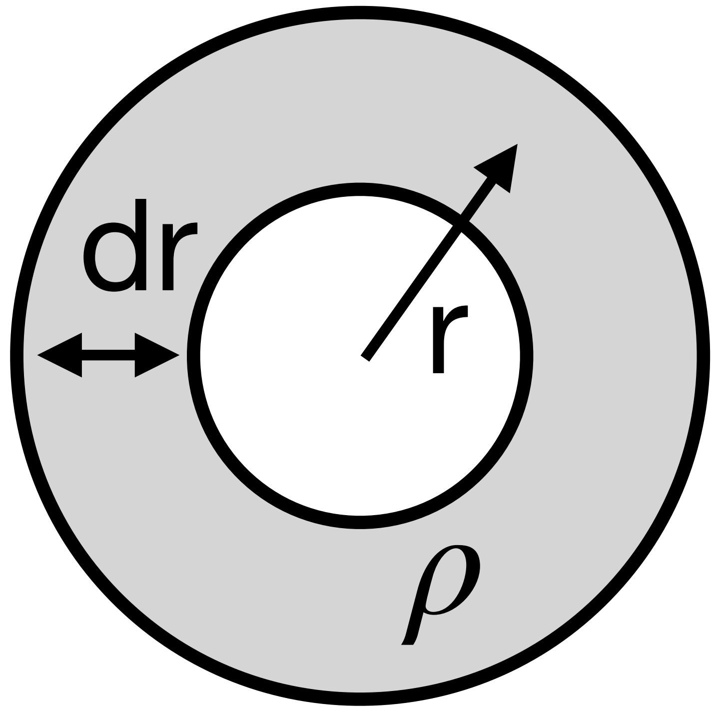
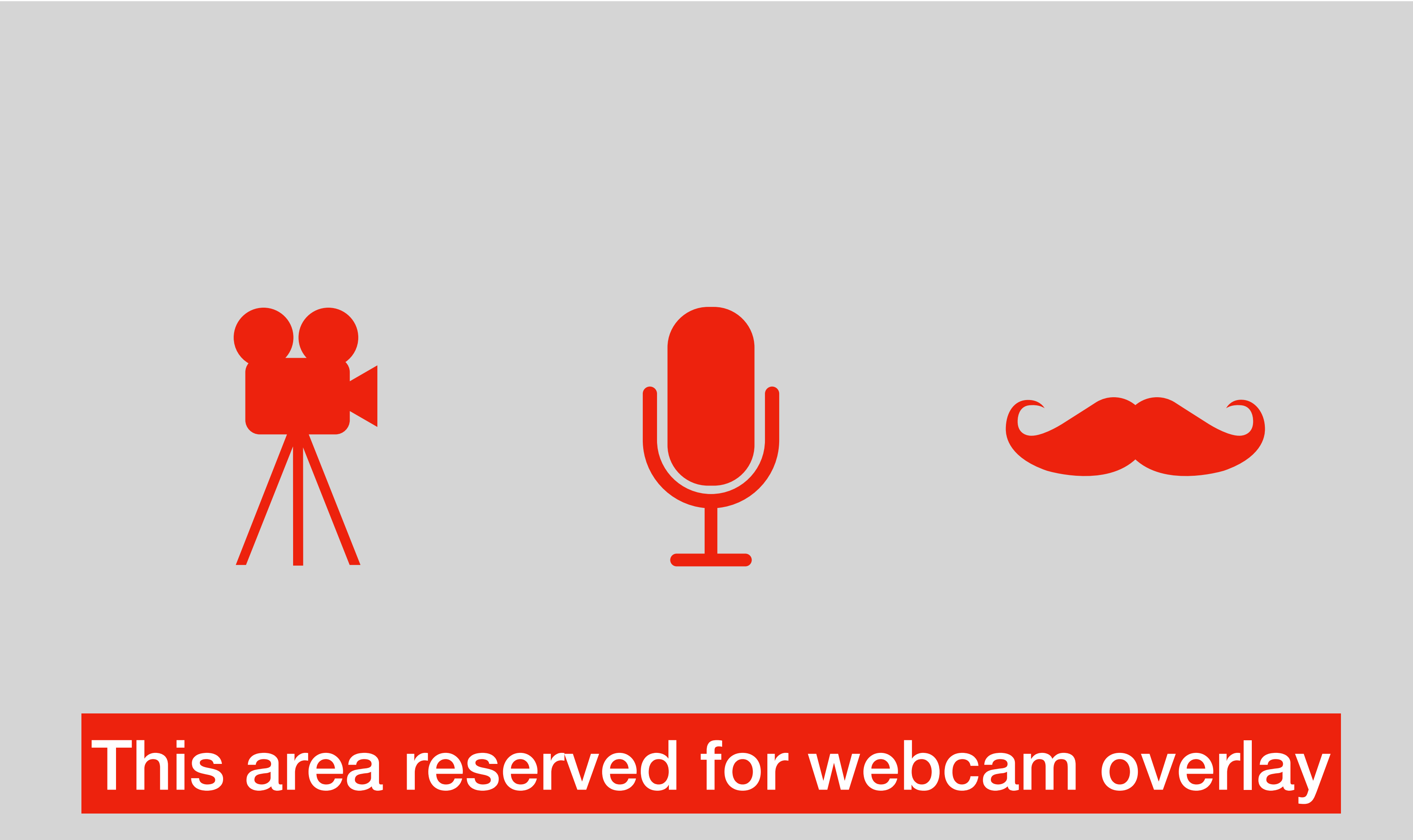
Stellar Structure Equations

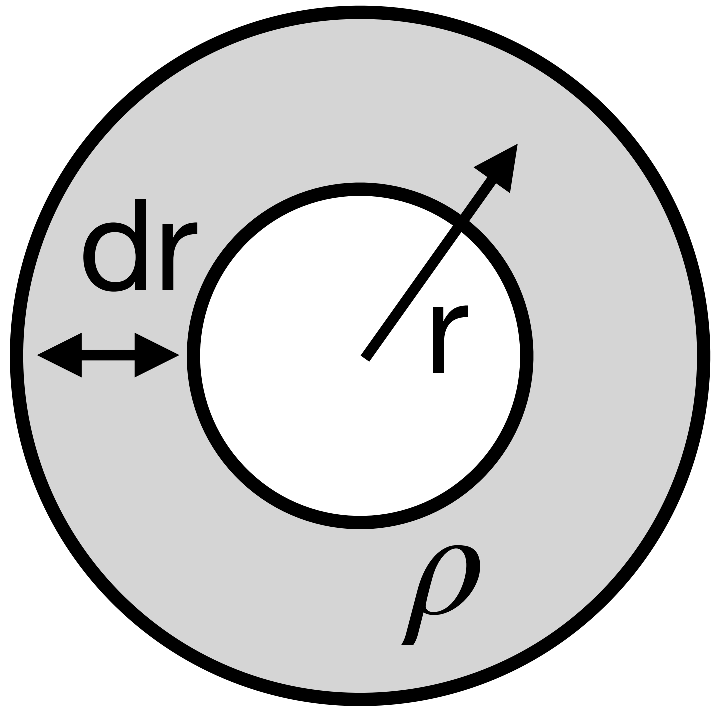


- You'll see these written as either a function of Radius or Mass
  - $dm = 4\pi r^2 \rho \, dr$
  - <u>I'll just focus on eqns as a function of Radius for now (as in BOB Ch 10)</u> This is called the "Eulerian form"… but the "Lagrangian form" is perhaps more elegant, since stars don't change M much over their lives, but DO change R (as we've already seen!)
  - I'm not going to derive these all for you here, *do* read through Ch 10 of BOB!



1. Mass Conservation

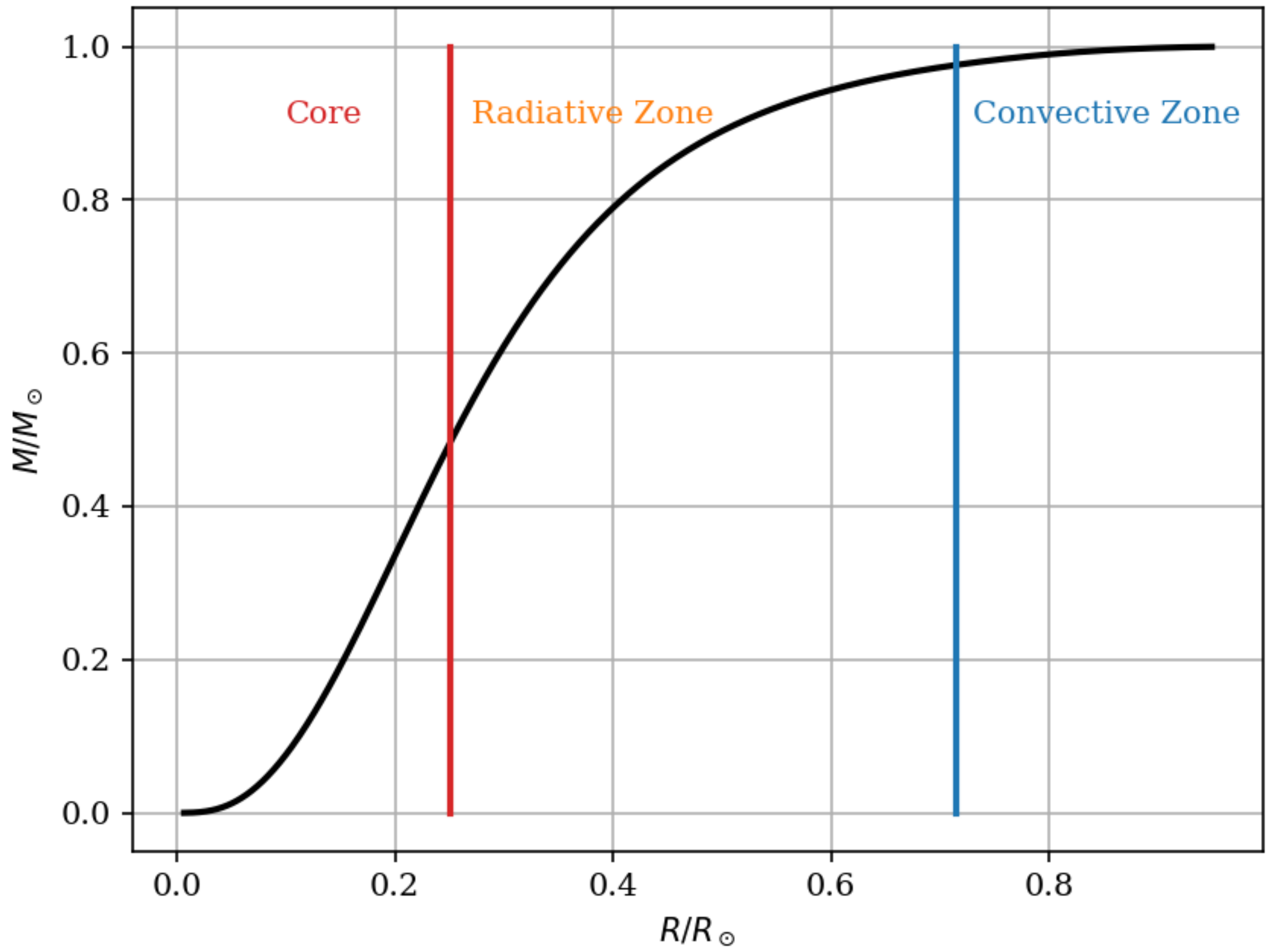
$$dm = 4\pi r^2 \rho \, dr$$

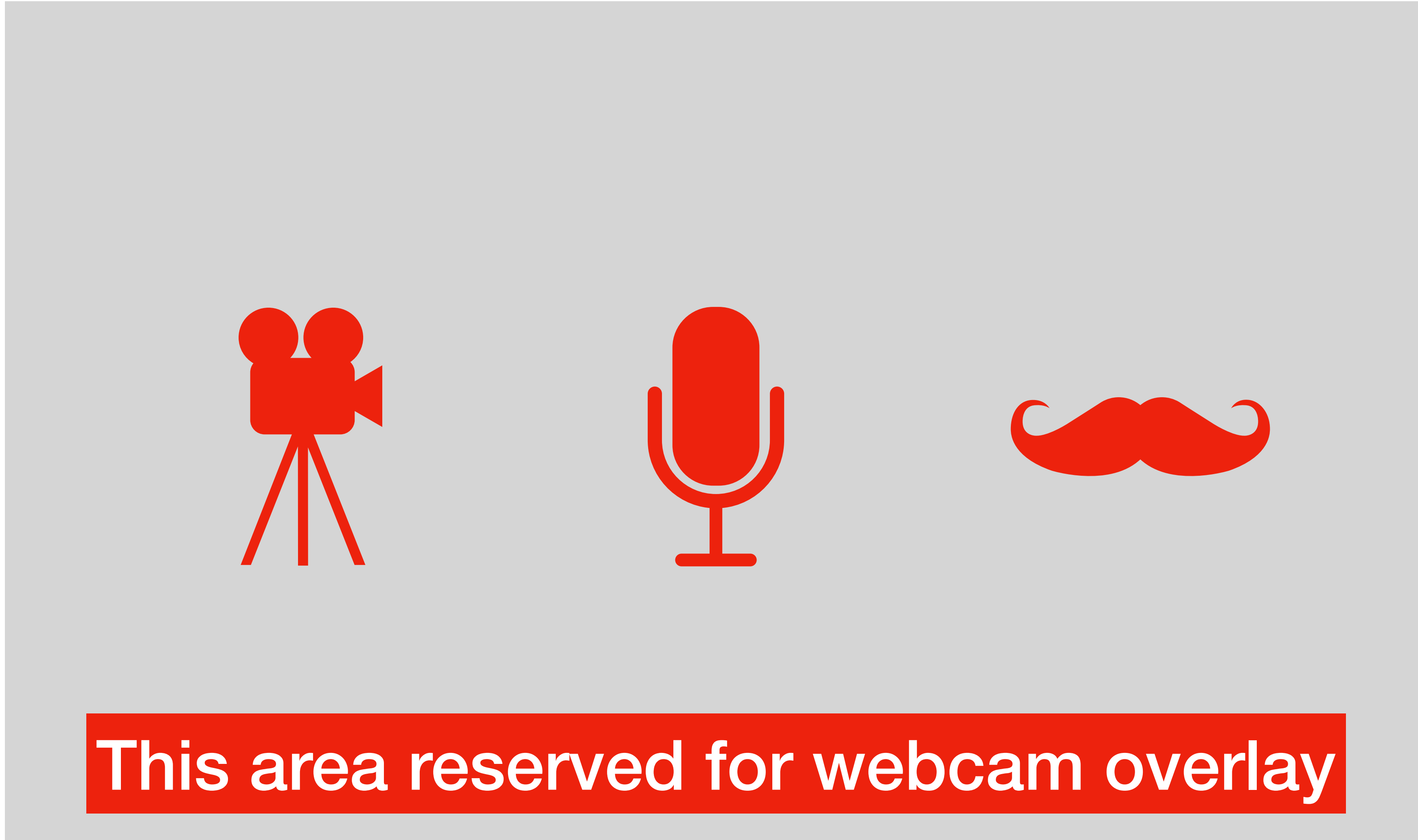


- Simple enough… the mass in a shell of constant density.
- Usually you see this rewritten slightly:

$$\frac{dm}{dr} = 4\pi r^2 \rho$$

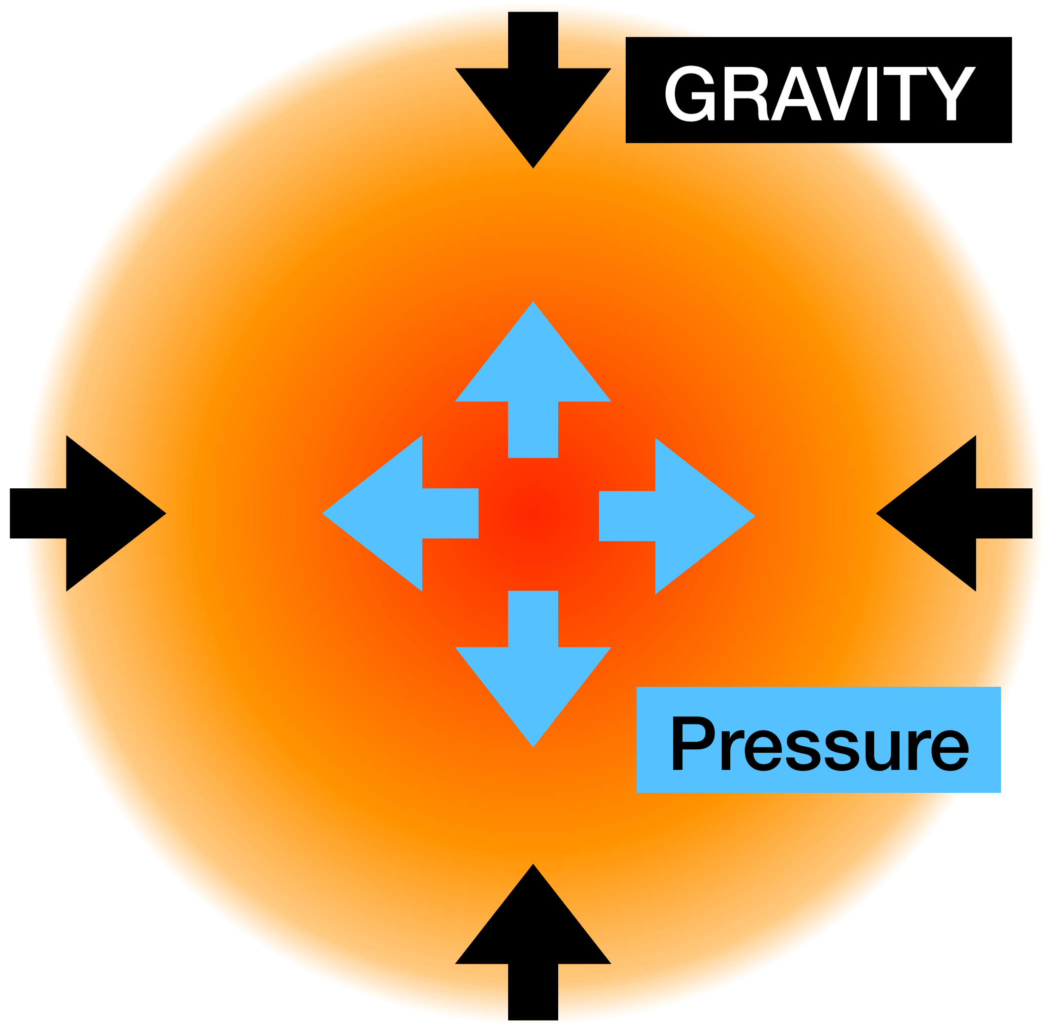
- Boundary Conditions at r=0 & r=R



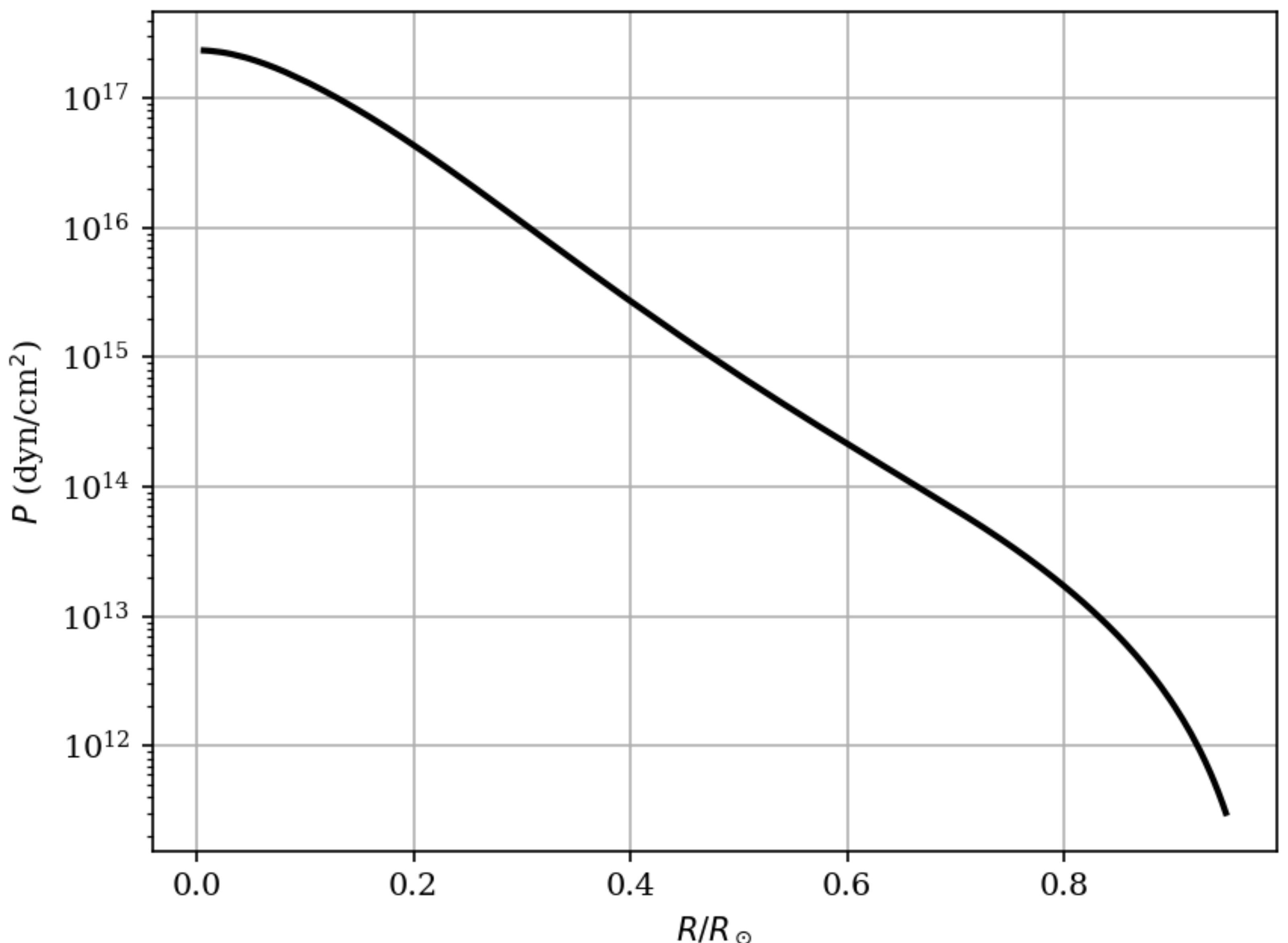


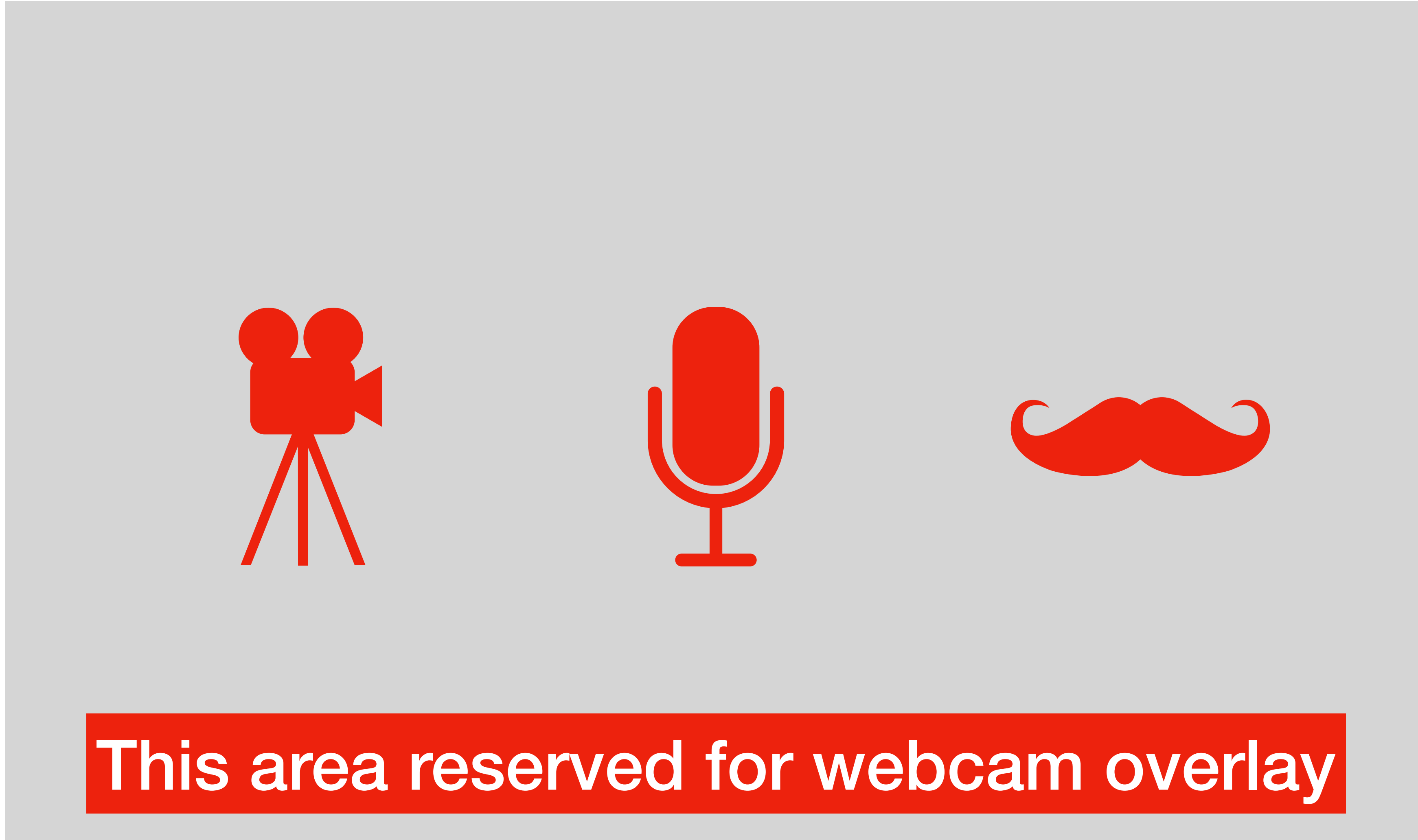
2. Hydrostatic Equilibrium

aka: Conservation of Momentum



- Star is static, acceleration throughout the interior must be ≈ 0
- $\dfrac{dP}{dr} = -G\dfrac{M_r \rho}{r^2} = -\rho g,$ where $g = GM_r/r^2$ is acceleration
- The change in pressure with radius (aka the pressure gradient) must balance the inward force of gravity



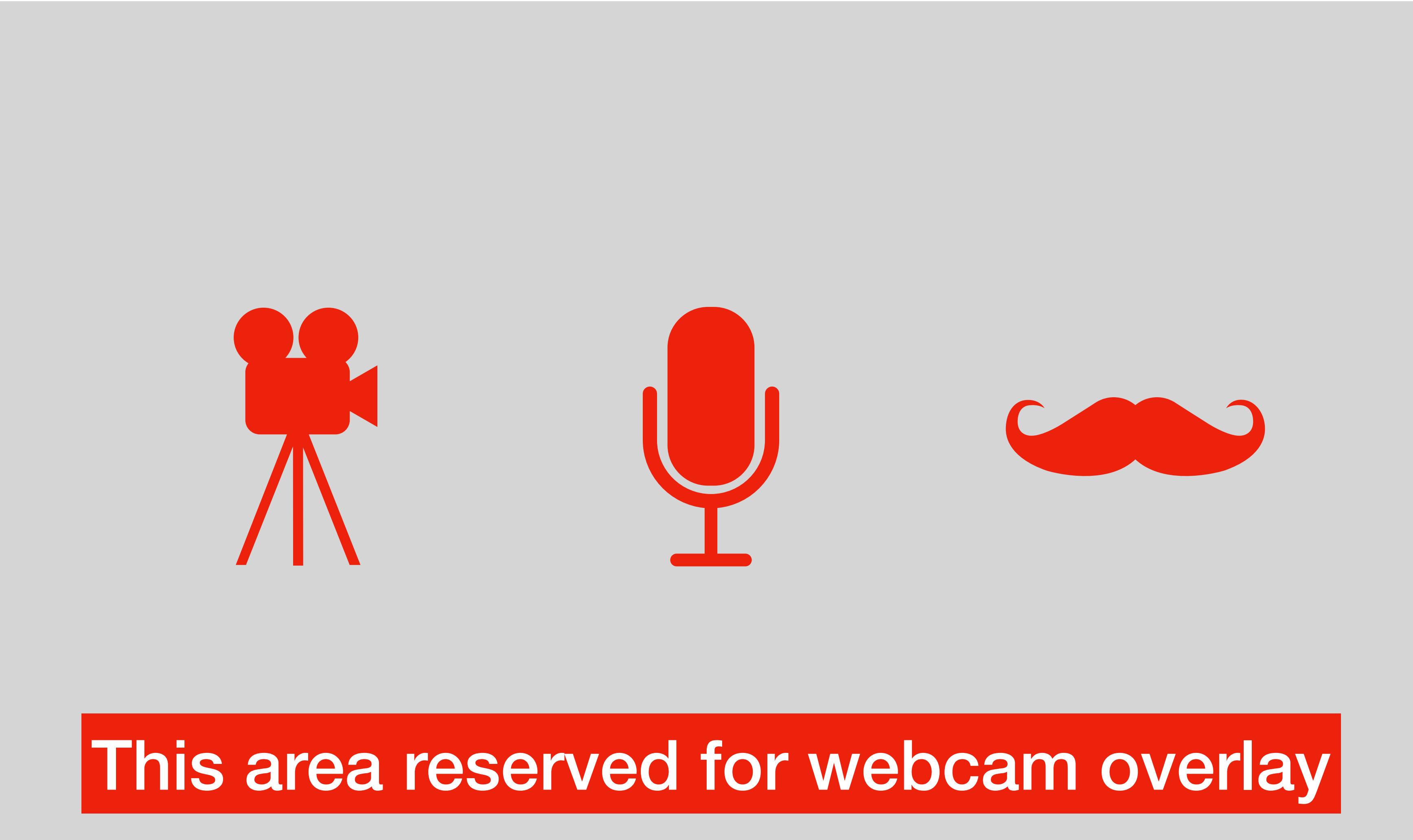


2. Hydrostatic Equilibrium

aka: Conservation of Momentum

- $\dfrac{dP}{dr} = -G\dfrac{M_r \rho}{r^2} = -\rho g$, where $g = GM_r/r^2$ is acceleration
- Interestingly, this can also be written as:

$\dfrac{dP}{d\tau} = \dfrac{g}{\bar{\kappa}}$, if we think back to Lecture 05, where we defined $d\tau = -\kappa\rho\, dr$



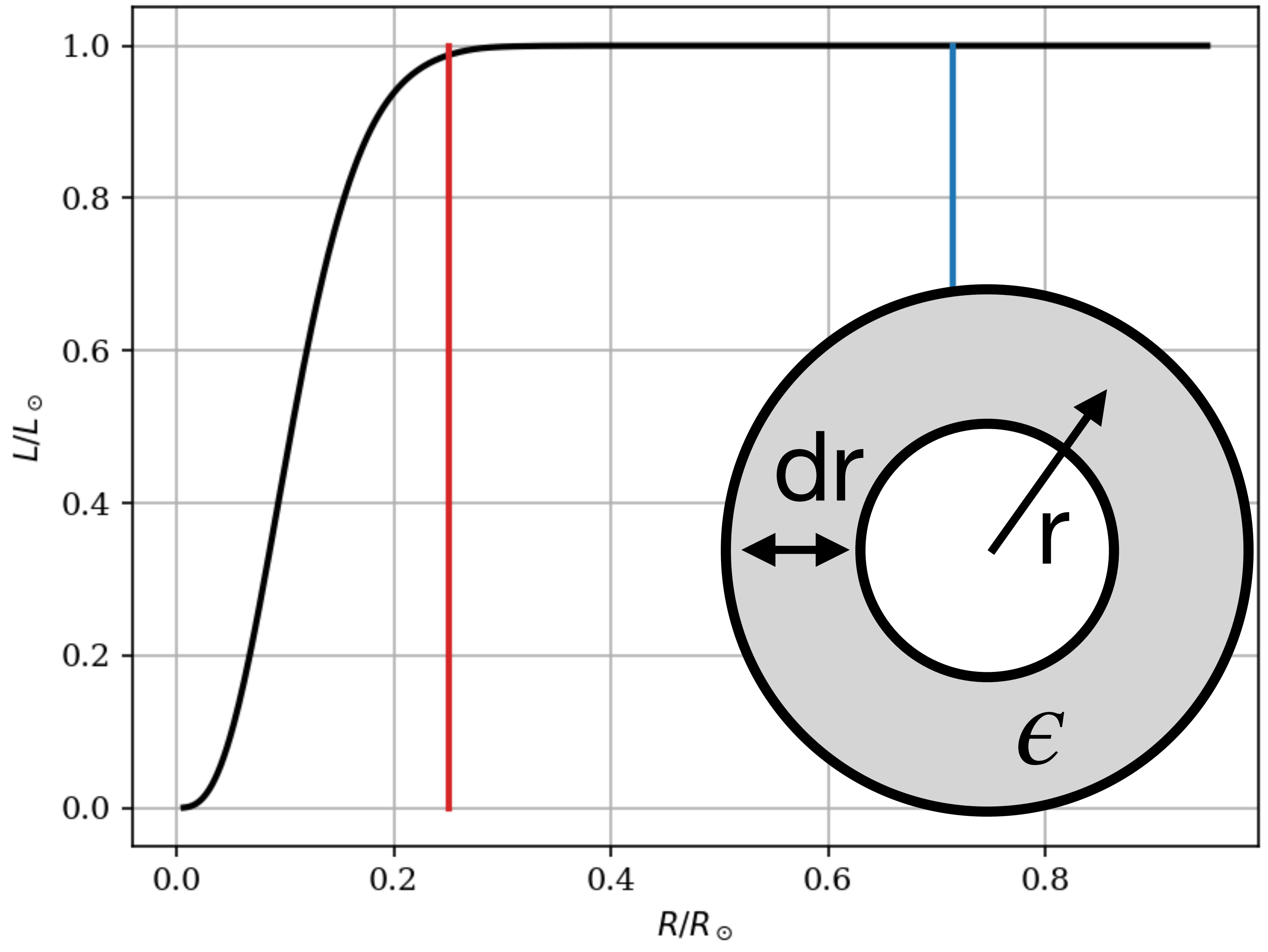
3. Energy Conservation

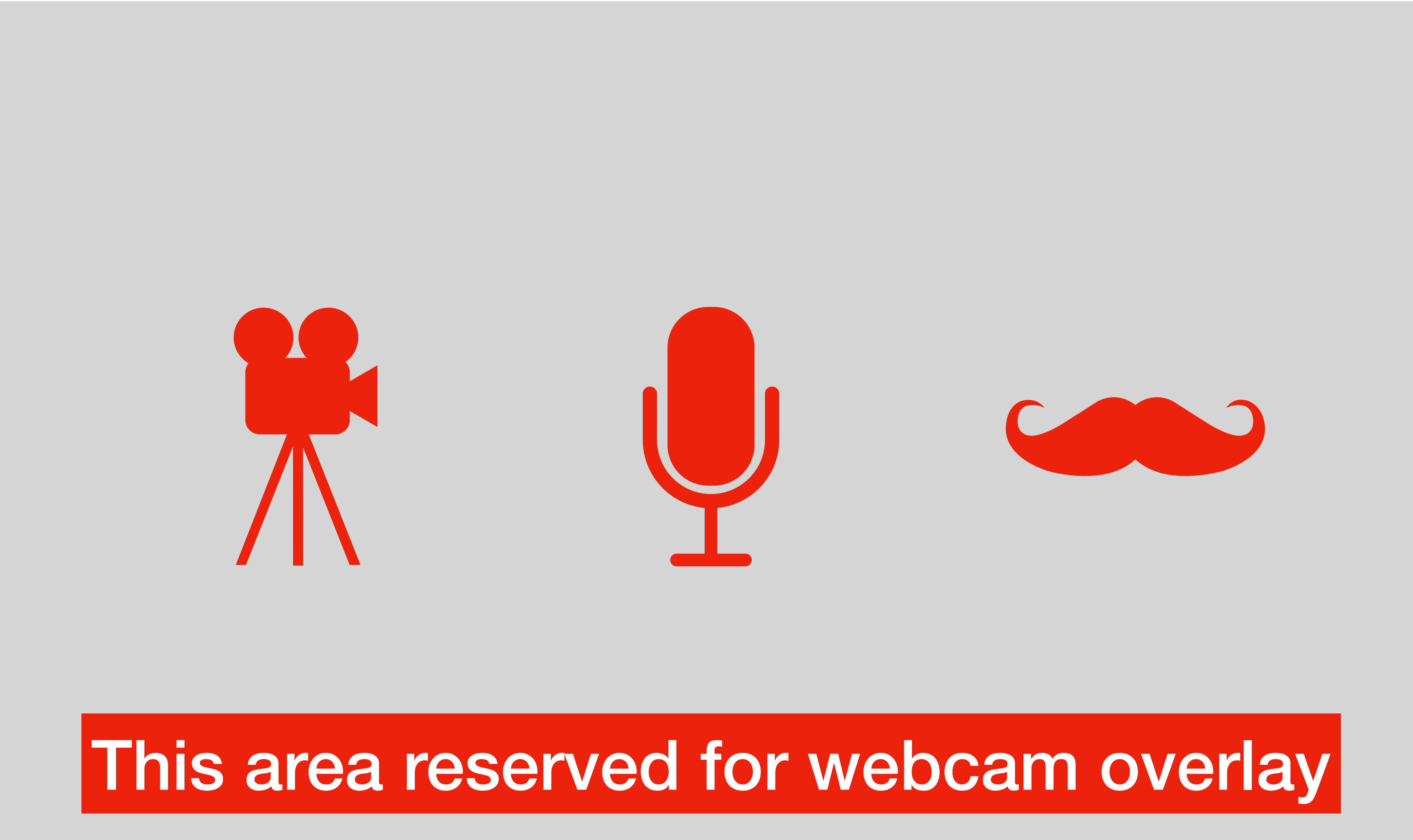
- All the light that shines into a layer of the star must shine out (unless light is *created:* the core)

- This can be written like mass conservation:
  $\frac{dL}{dr} = 4\pi r^2 \rho\ \epsilon$, where ϵ is the energy released

  (generated through fusion, neutrinos, or gravity)

- This “luminosity gradient” is flat everywhere except the core,





4. Energy Transport Equation(s)

- This one is much harder… how does energy move through the star radially?
- <u>3 possible transport mechanisms:</u>
  Conduction (not important for dwarf or giants, but matters for e.g. white dwarfs)
  Radiation (we've discussed this lots! Opacity important)
  Convection (remember the old pot of boiling water)
- Each mechanism has different solution to the relevant differential equation: $\dfrac{dT}{dr}$
- The "temperature gradient"

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4. Energy Transport Equation(s)

- For **radiative transport**, the temperature gradient is:

$$\frac{dT}{dr} = \frac{3\bar{\kappa}\rho L}{64\pi r^2 \sigma T^3}$$

- This has lots of pieces that are *familiar…* can <u>almost</u> re-write as “flux transport”:

- $$dF = \frac{L(\bar{\kappa}\rho)dr}{4\pi r^2} = \sigma\,(T^3 dT)$$

Missing a constant still

Recall: $l = 1/\kappa\rho$

basically T^4

- And several other ways of writing this that I don’t find any more *intuitive*

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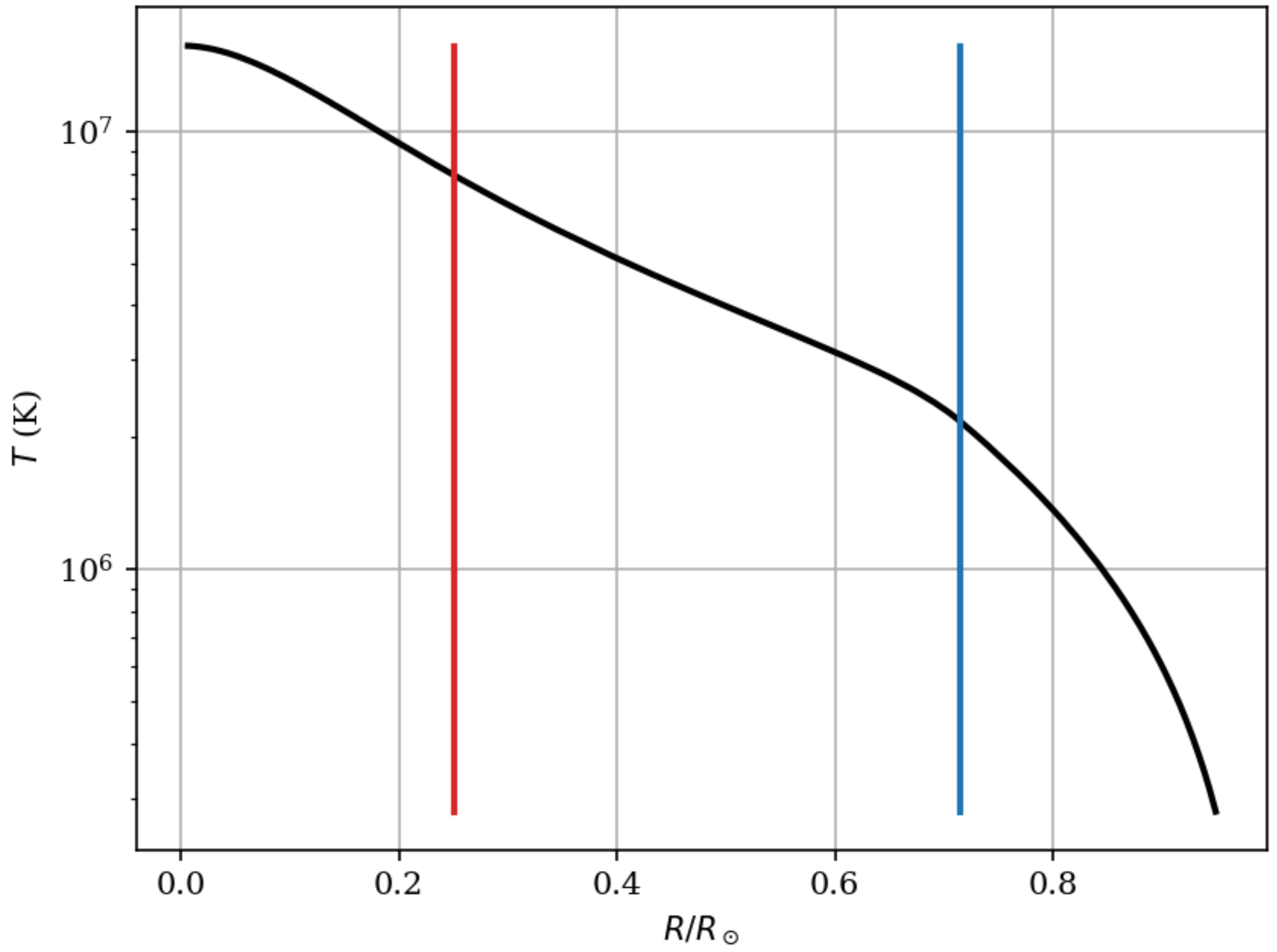
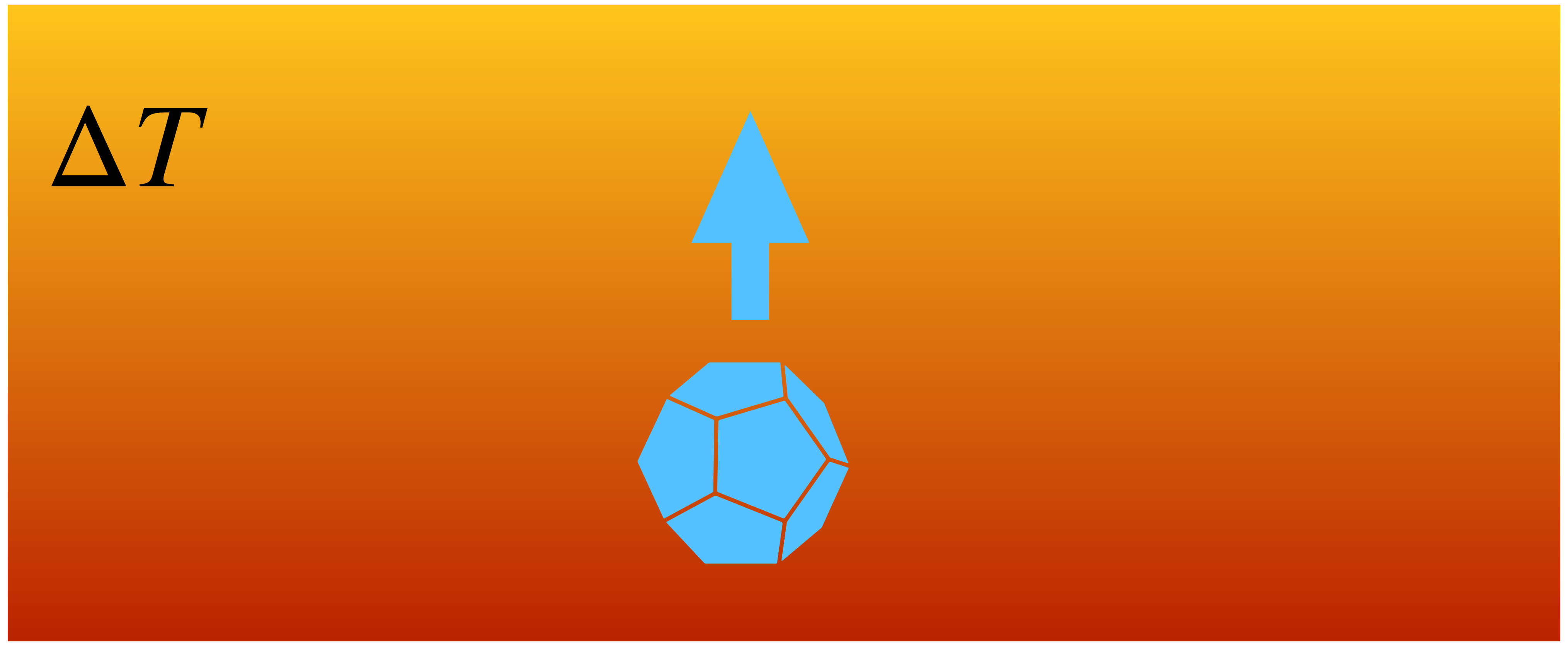
4. Energy Transport Equation(s)

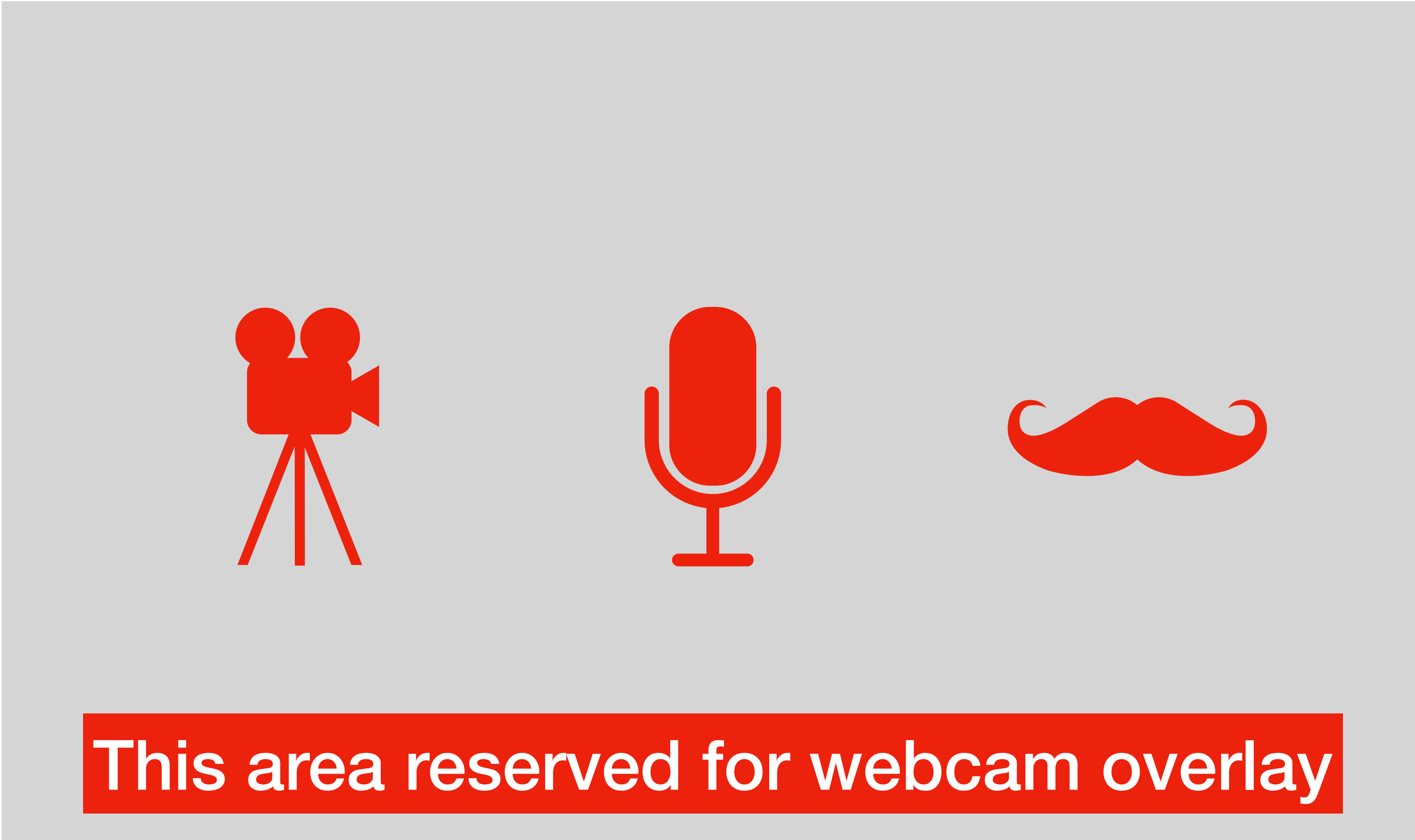
- Q: When does a star use **convective** instead of **radiative** energy transport?

- A: When the temperature gradient is high!

  i.e. when a blob of gas would become *buoyant* and rise faster than it could radiate energy away and come into LTE

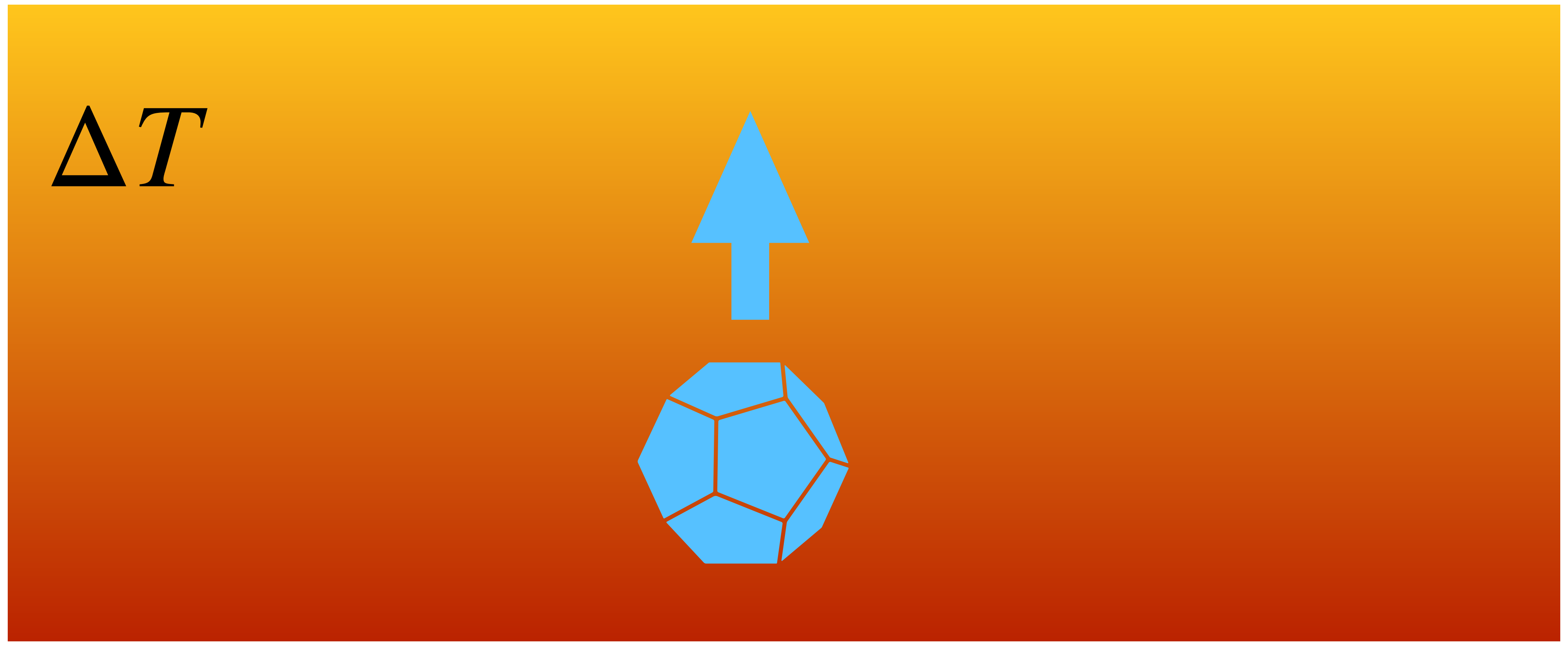


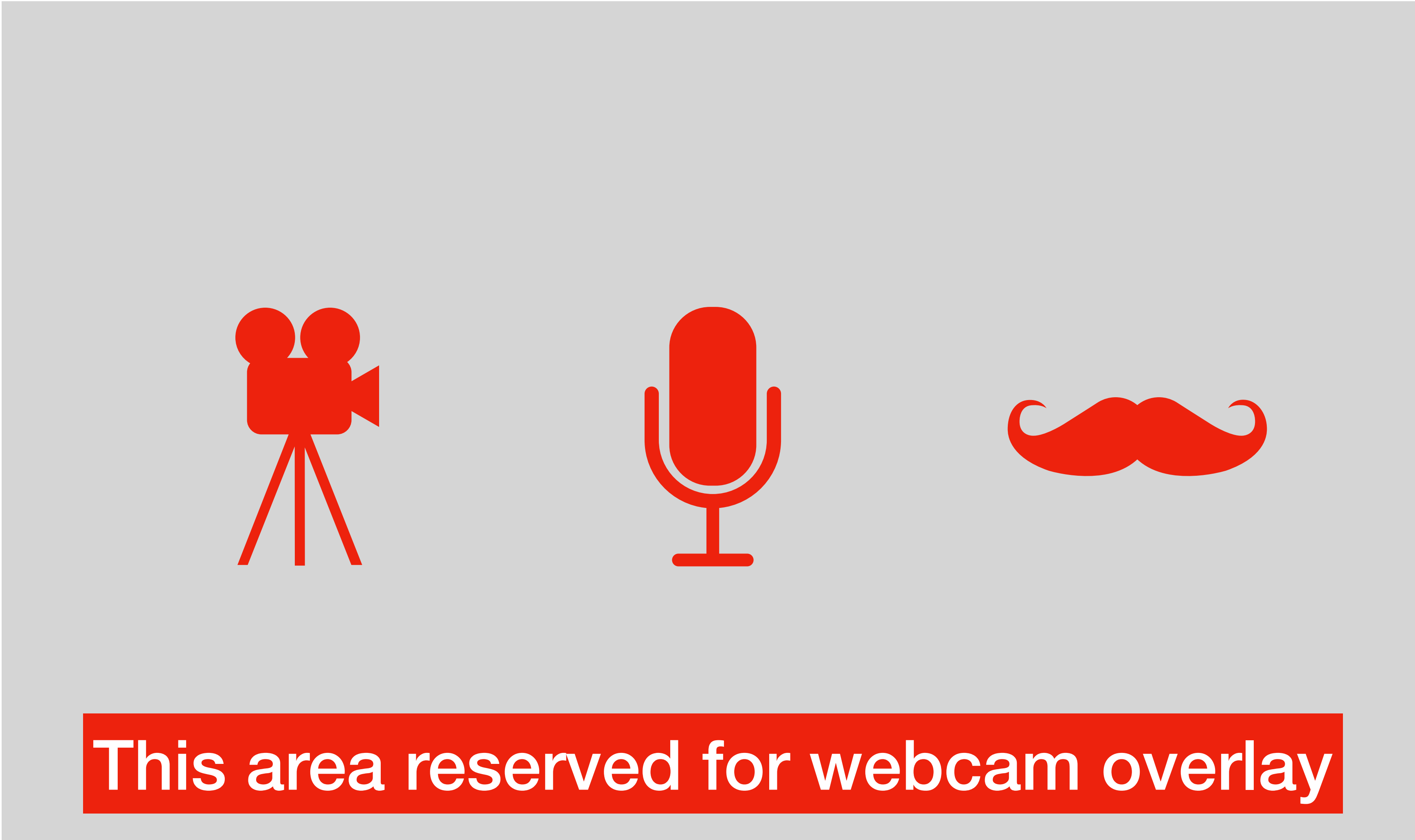




4. Energy Transport Equation(s)

- For **convective transport**, the temperature gradient is:
  $\frac{dT}{dr} = -\frac{g}{C_P}$, where C_P is the heat capacity of the gas (at constant pressure)
- As before, other ways of writing this, that aren't especially intuitive (to me)





Stellar Structure Equations

$$\frac{dP}{dr} = -G\frac{M_r \rho}{r^2} = -\rho g$$

$$\frac{dM}{dr} = 4\pi r^2 \rho$$

$$\frac{dL}{dr} = 4\pi r^2 \rho \, \epsilon$$

$$\frac{dT}{dr} = \frac{3\bar{\kappa}\rho L}{64\pi r^2 \sigma T^3} \qquad \frac{dT}{dr} = -\frac{g}{C_P}$$

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Estimate central pressure of a star

- Start w/ hydrostatic equilibrium $\frac{dP}{dr} = -\rho g$
- Assume star has a constant density $\bar{\rho} = M/V = 3M/4\pi R^3$
- If we adopt some boundary conditions: $\frac{dP}{dr} = \frac{P_s - P_c}{r_s - r_c} \approx \frac{P_c}{R}$
- Then we can solve for $P_c = \frac{3GM^2}{4\pi R^4}$
- Roughly 3×10^{15} dyne/cm^2

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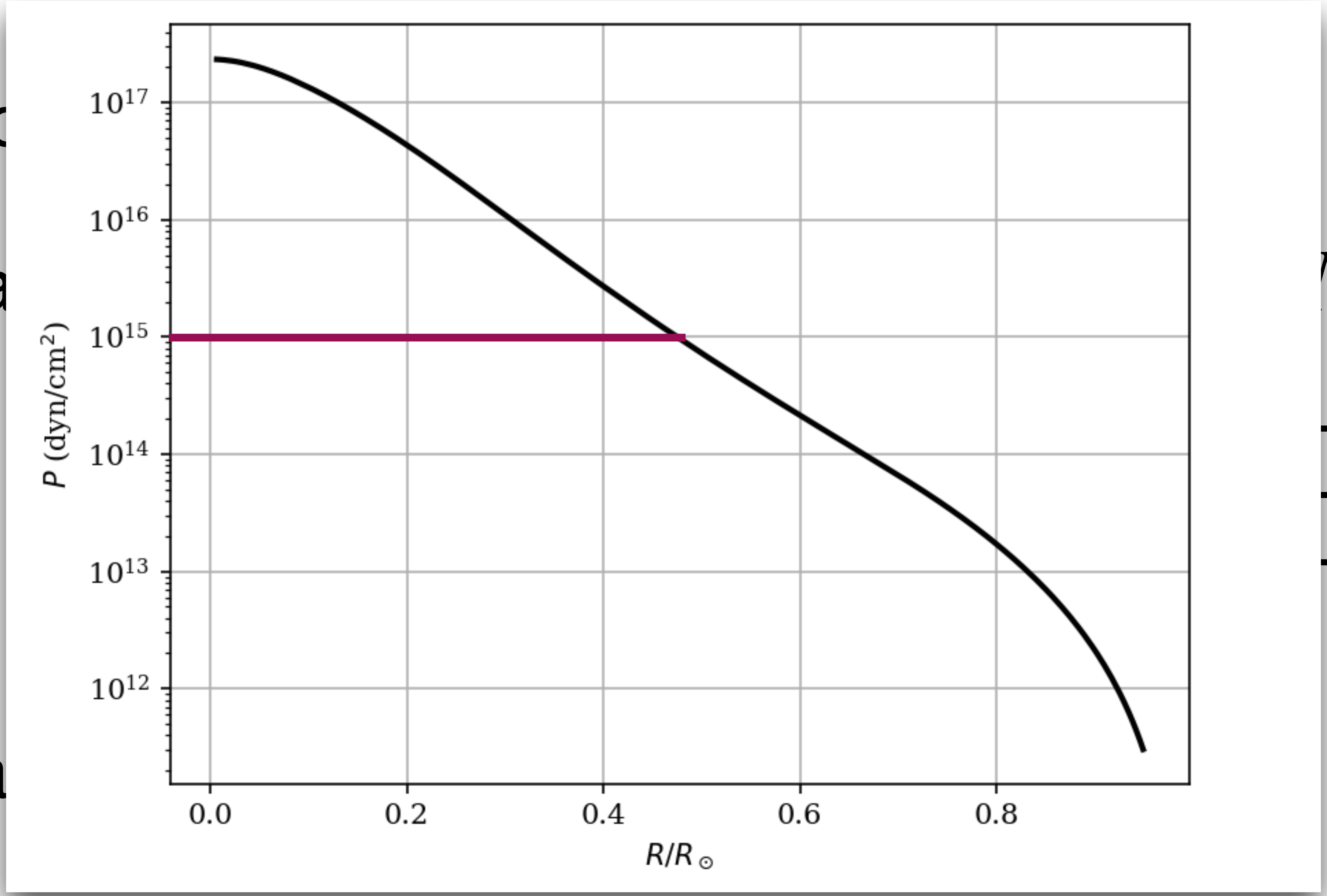
Estimate central pressure of a star

- Start w/ hyd

- Assume sta $M/4\pi R^3$

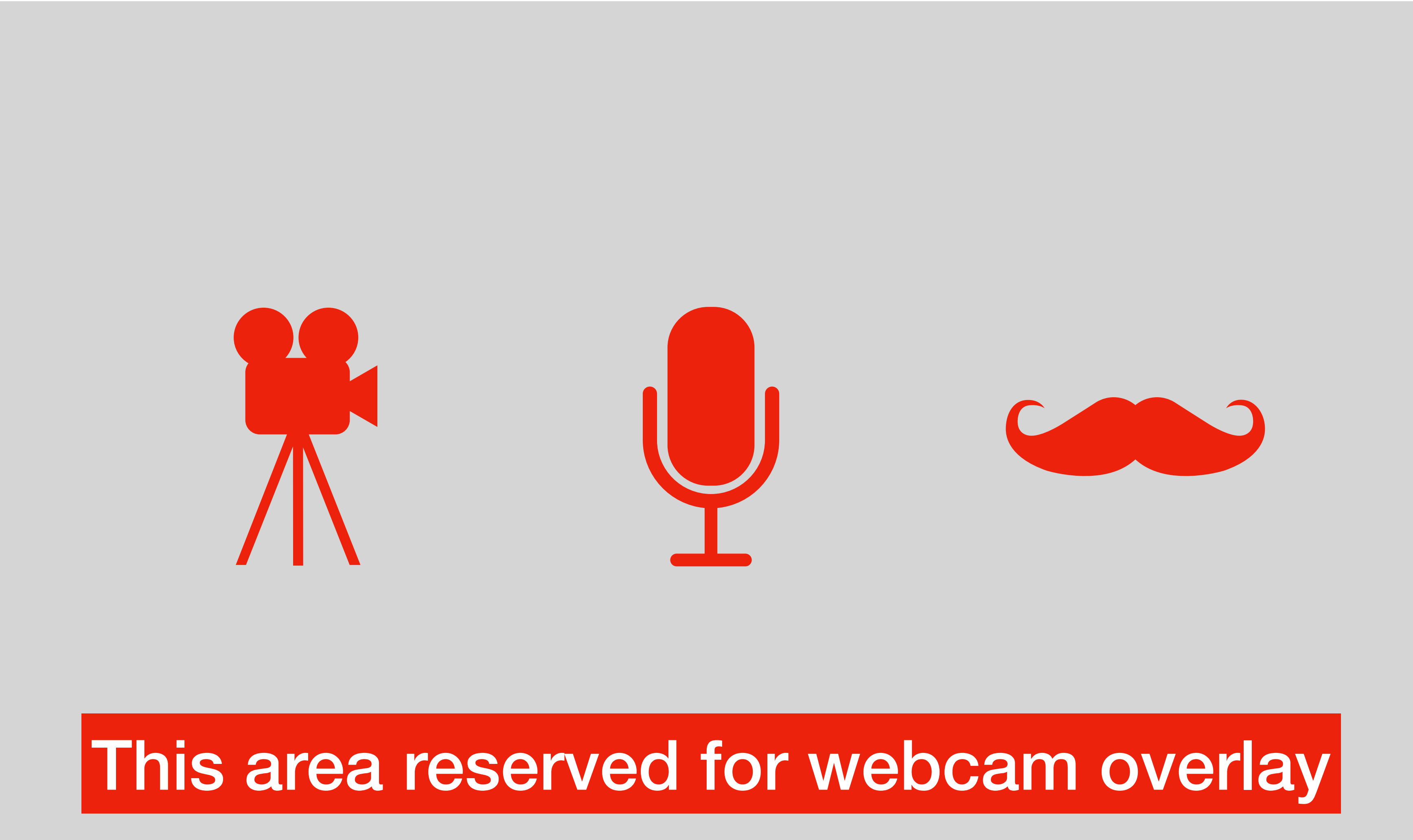
- If we adopt $- P_c$ / $- r_c$ $\approx \frac{P_c}{R}$

- Then we ca



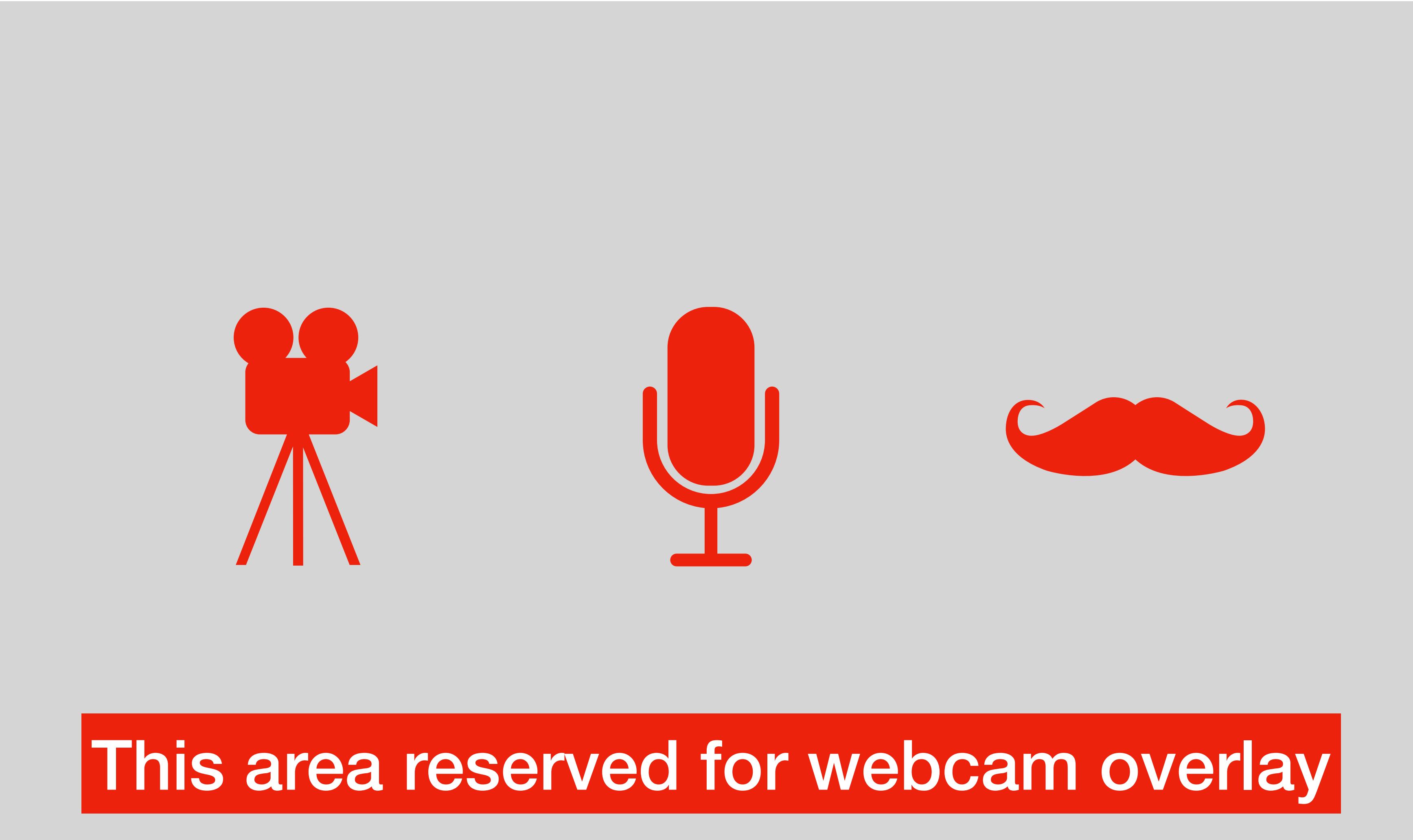
- Roughly 3×10^{15} dyne/cm^2

Not super accurate, but informative

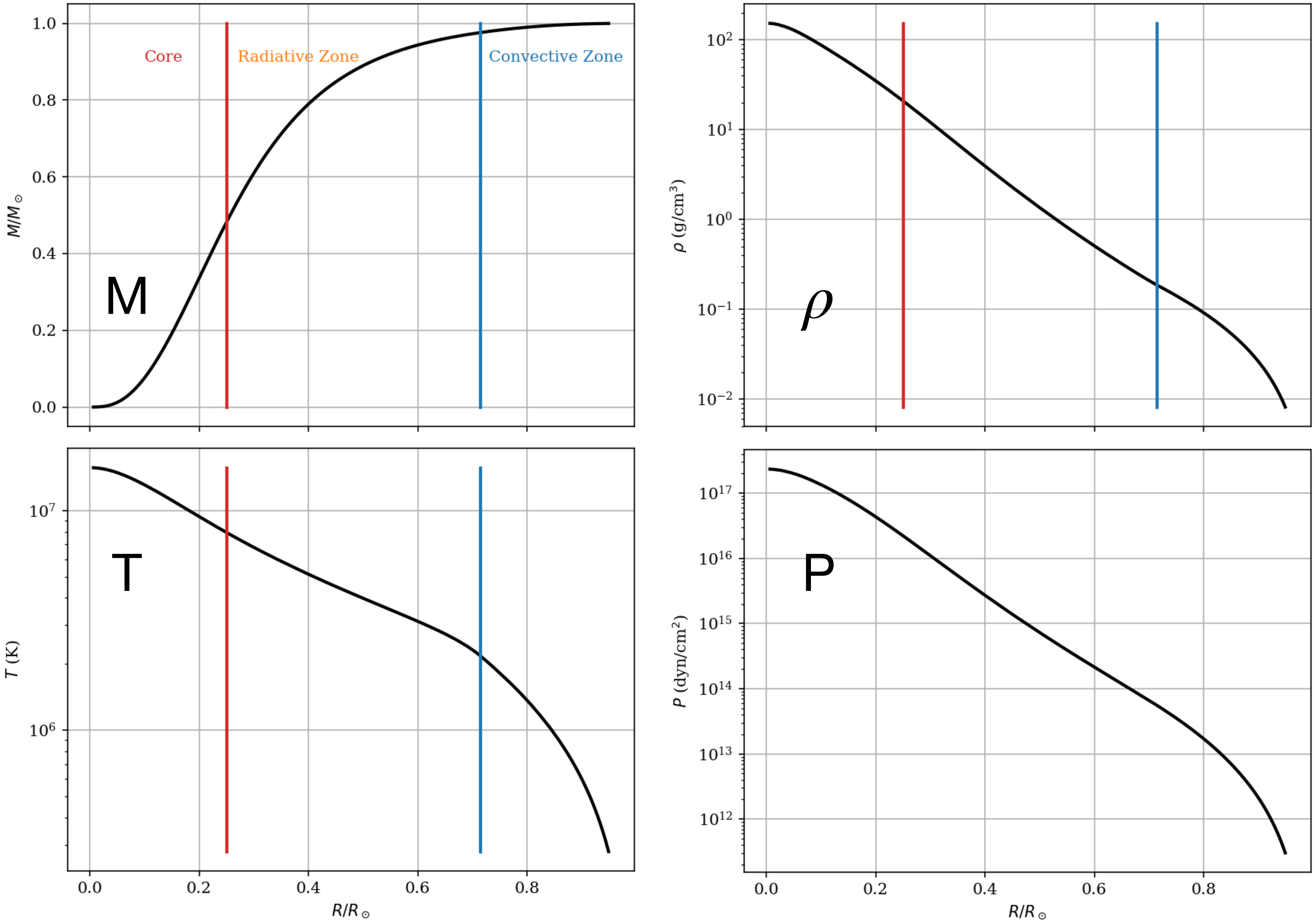


Equation of State (EOS)

- So… a constant density of gas is *probably* not realistic for most stars!
- The EOS connects "state variables" for a gas, such as T, ρ, P, V
- You're possibly familiar w/ EOS for an ideal gas, comes in forms like: $PV = nRT = Nk_BT$, can make various substitutions based on type of gas or container or experiment…



Stellar Structure Equations



A few of these curves look *very* similar….

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Polytropes and the EOS

- A Polytrope is a self-gravitating sphere, where hydrostatic equilibrium is at work
- $$\boxed{P \propto \rho^{\gamma}, \text{ where } \gamma = \frac{1+n}{n}}$$
- n is called the “polytropic index”
- Can also be written as $pV^{n} = \text{Const}$
- Polytropes *can* be an Equation of State solution!

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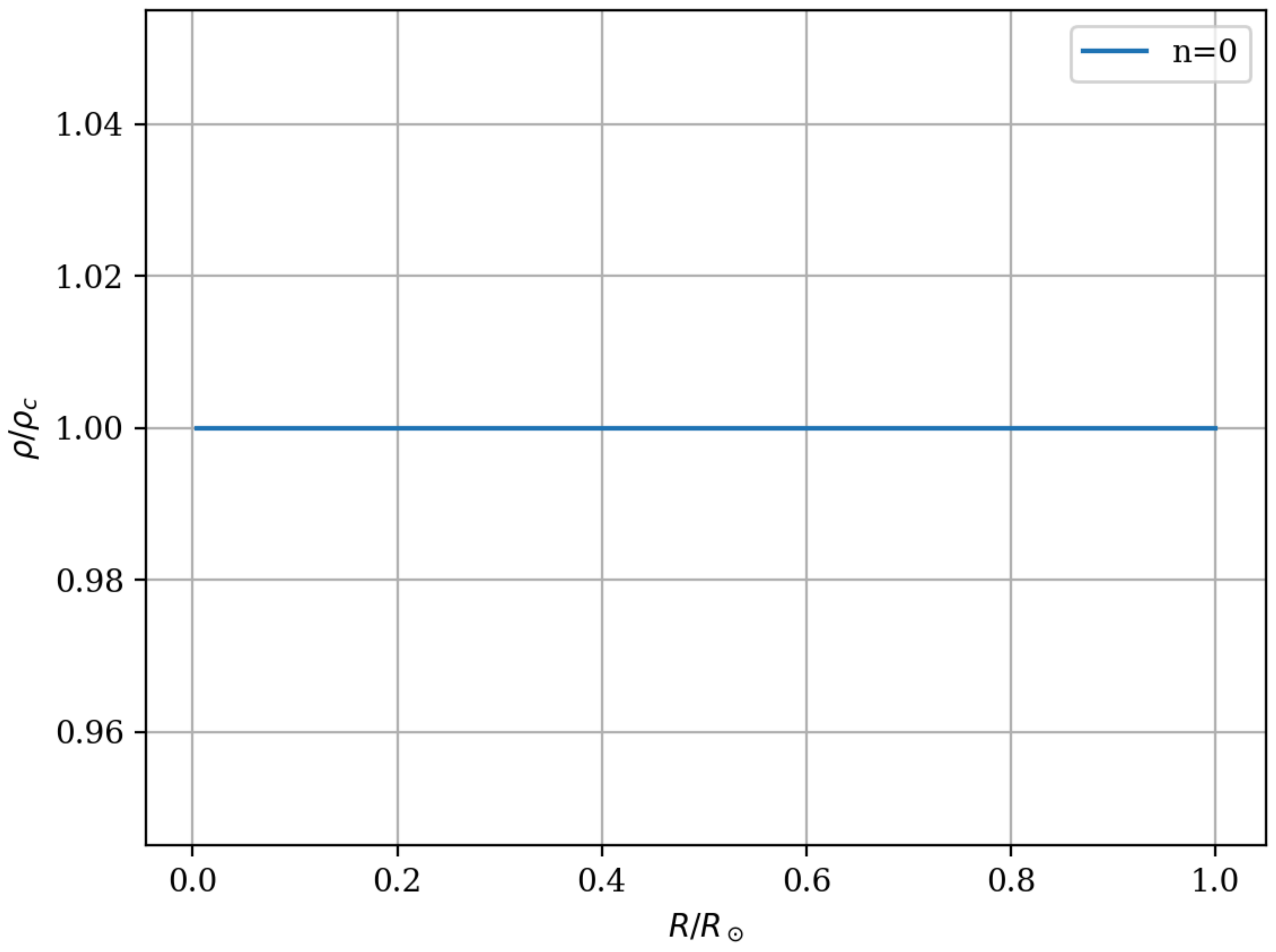
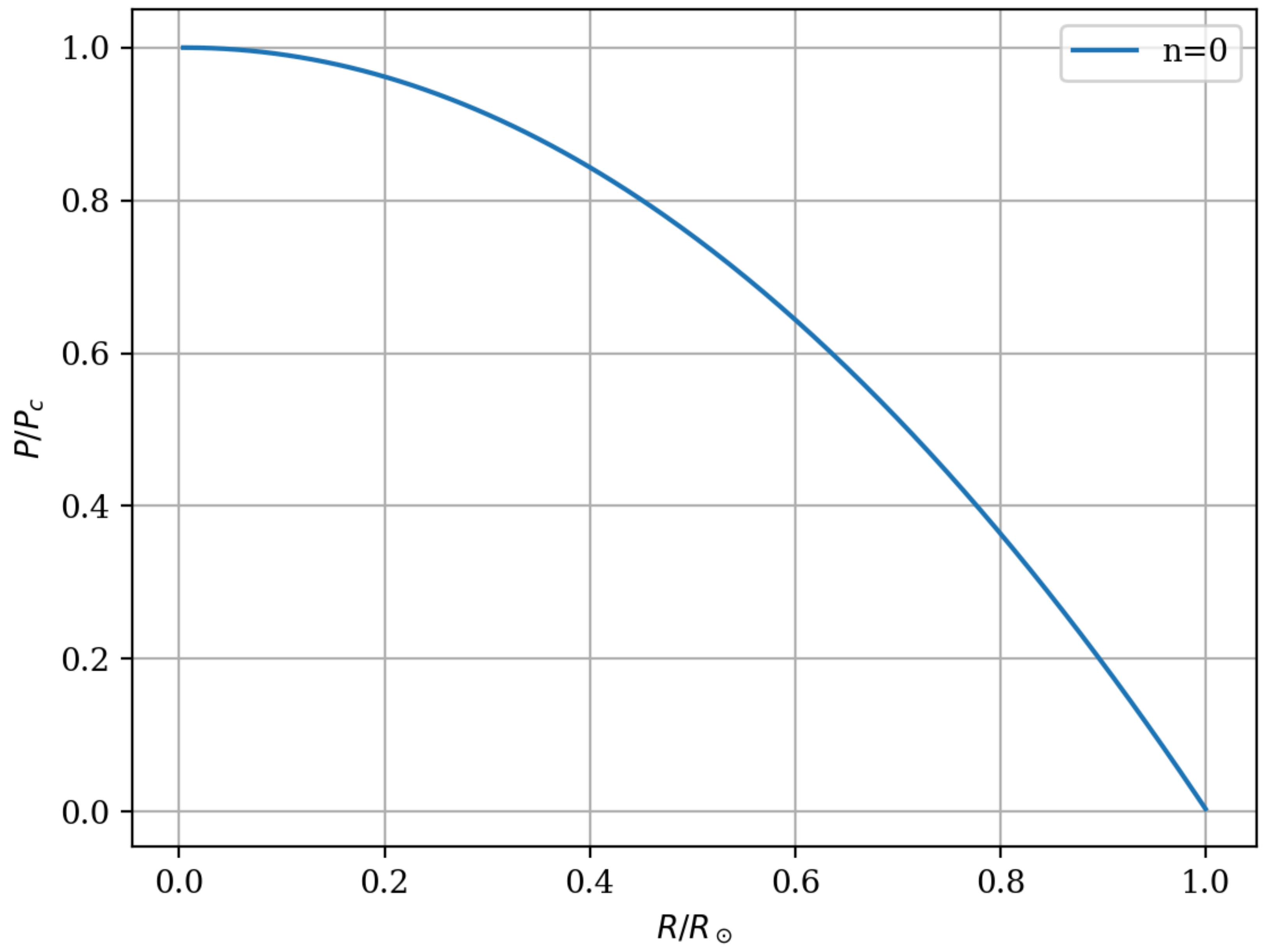
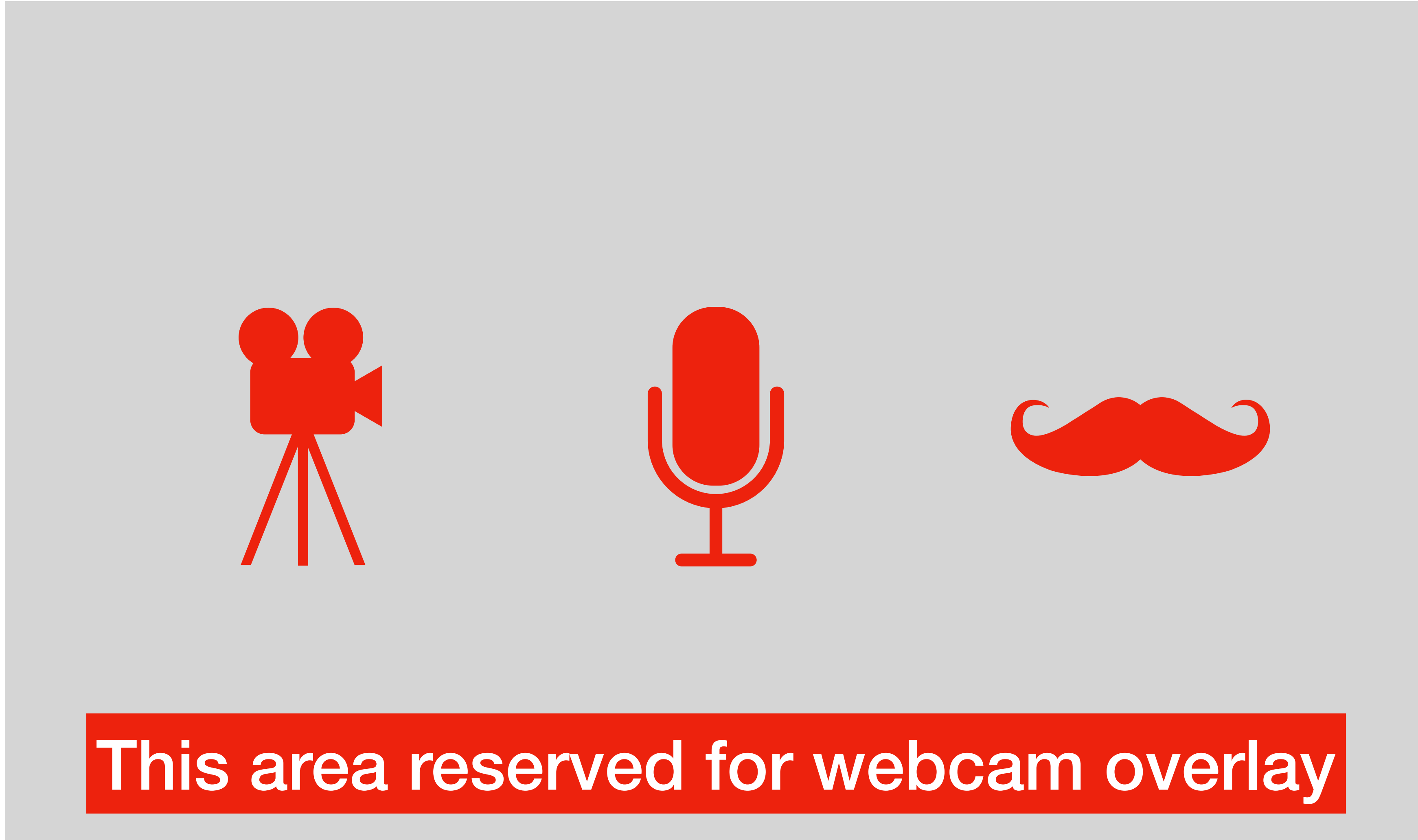
Polytropes and the EOS

$P \propto \rho^{(1+n)/n}$

- One interesting solution is $n = 0$ ("isobaric", constant pressure)
- In this case, a *constant density sphere*
  - A crude approximation of a rocky (incompressible) planet



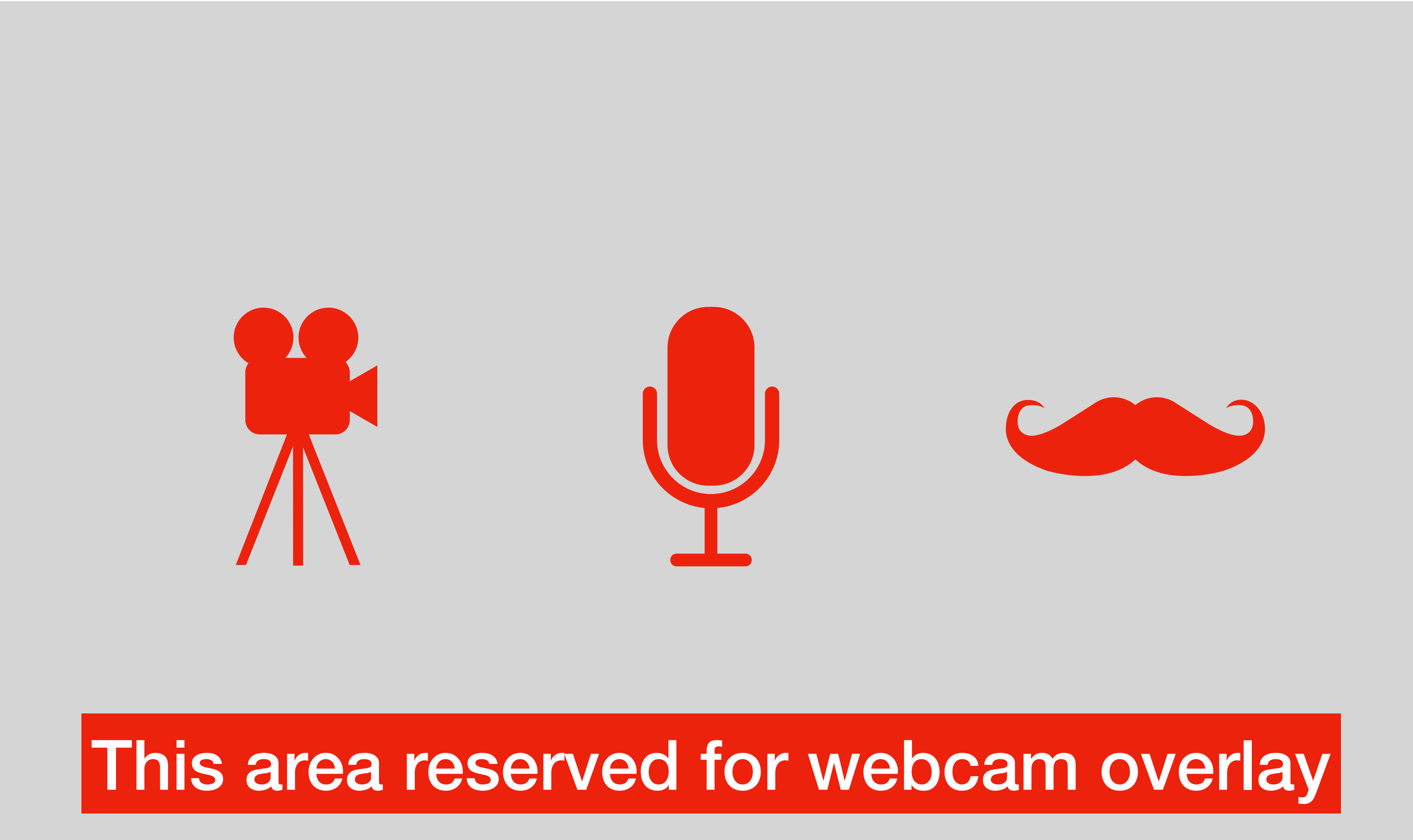




Polytropes and the EOS

$P \propto \rho^{(1+n)/n}$

- Q: So what do we *do* with them?
- A: we typically use polytropes to describe (approximate) the density and pressure structure throughout a star.
  - These are NOT proper stellar models, nor solutions to the complexities of the EOS!
- **Higher n: density more heavily weighted towards the center!**
- Typically in astronomy, a polytrope is a *solution* to the **Lane-Emden Equation**



Lane-Emden Equation

- Start w/ HSE: $$\frac{dP}{dr} = -G\frac{M_r \rho}{r^2}$$
- Take derivative with radius, and substitute in mass conservation: $$\frac{dM}{dr} = 4\pi r^2 \rho$$
- $$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho$$
- This describes the state of the self-gravitating star w/o any knowledge of radiation or transport

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Lane-Emden Equation

- $\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho$
- Now make a couple of (somewhat opaque) substitutions
  - $\rho/\rho_c = \theta^n$ (polytropic temperature), $\xi = r/\alpha$ (scale radius)
  - and since these are polytropes, density and pressure are connected: $P/P_c = \theta^{n+1}$

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Lane-Emden Equation

- $\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho$ … now becomes:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n$$

$$\rho/\rho_c = \theta^n$$

$$P/P_c = \theta^{n+1}$$

$$\xi = r/\alpha$$

- This "simply" solves the density and pressure structure of the star with only 1 free parameter: n, which describes the central concentration of density

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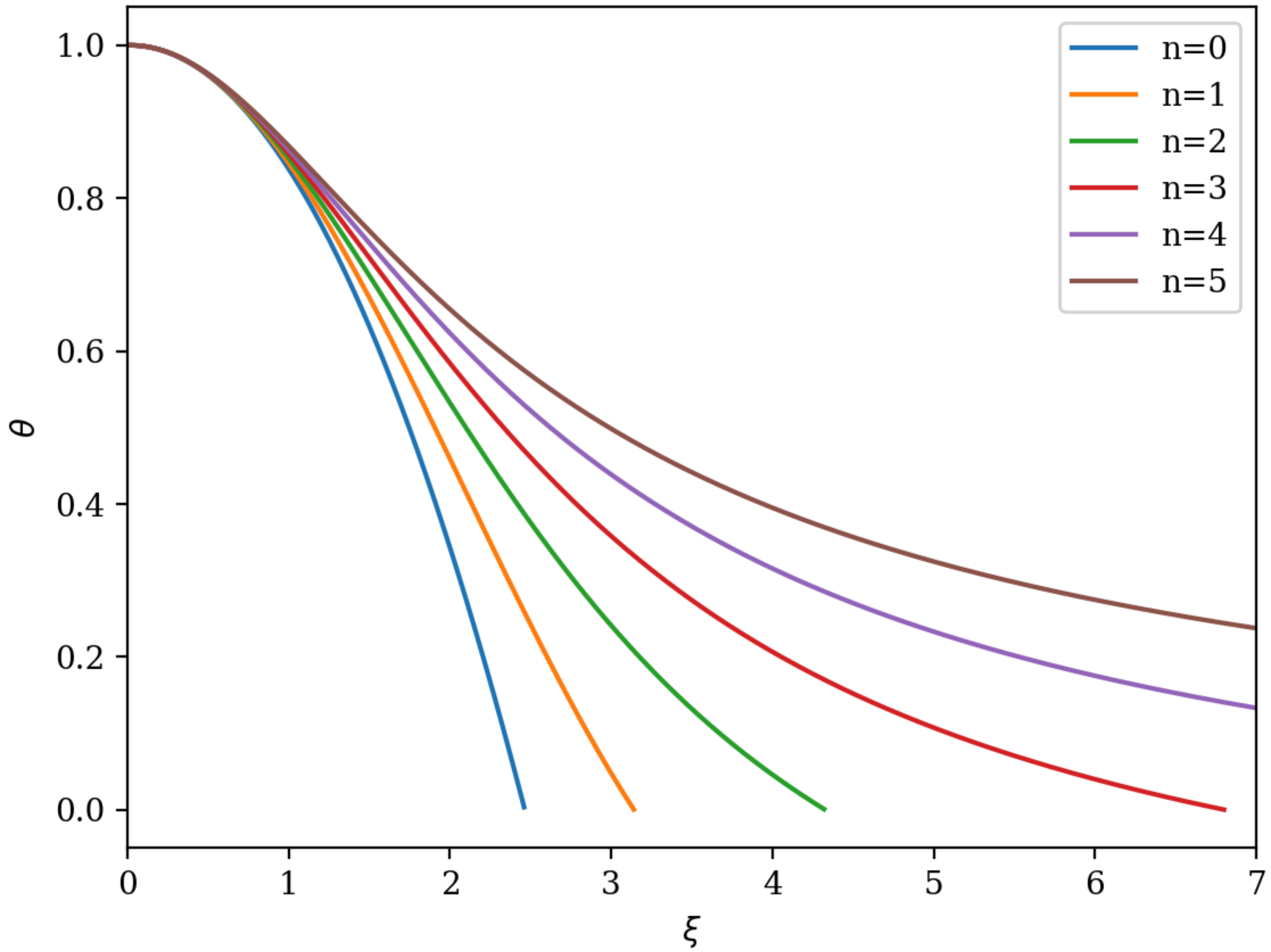
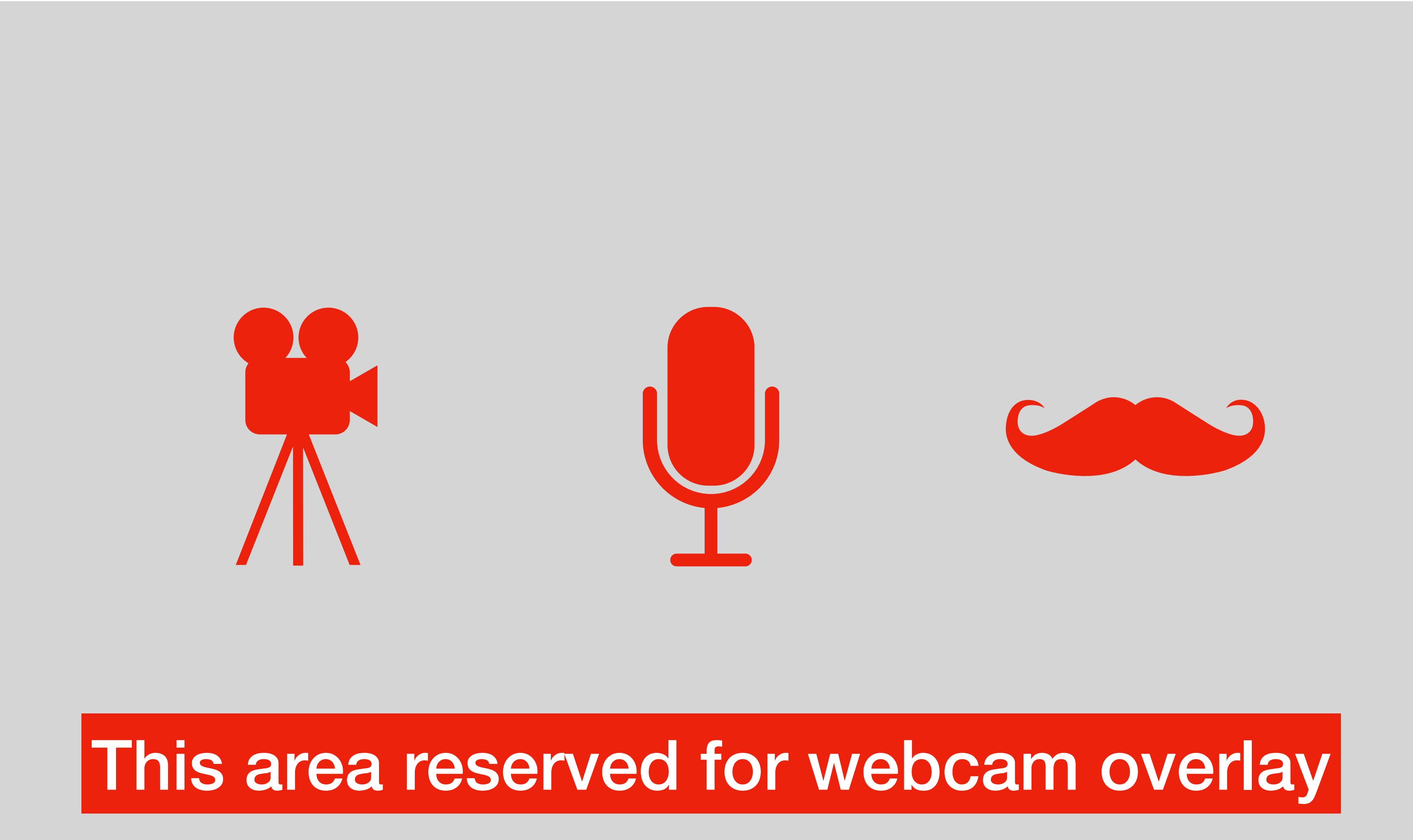
Lane-Emden Equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n$$

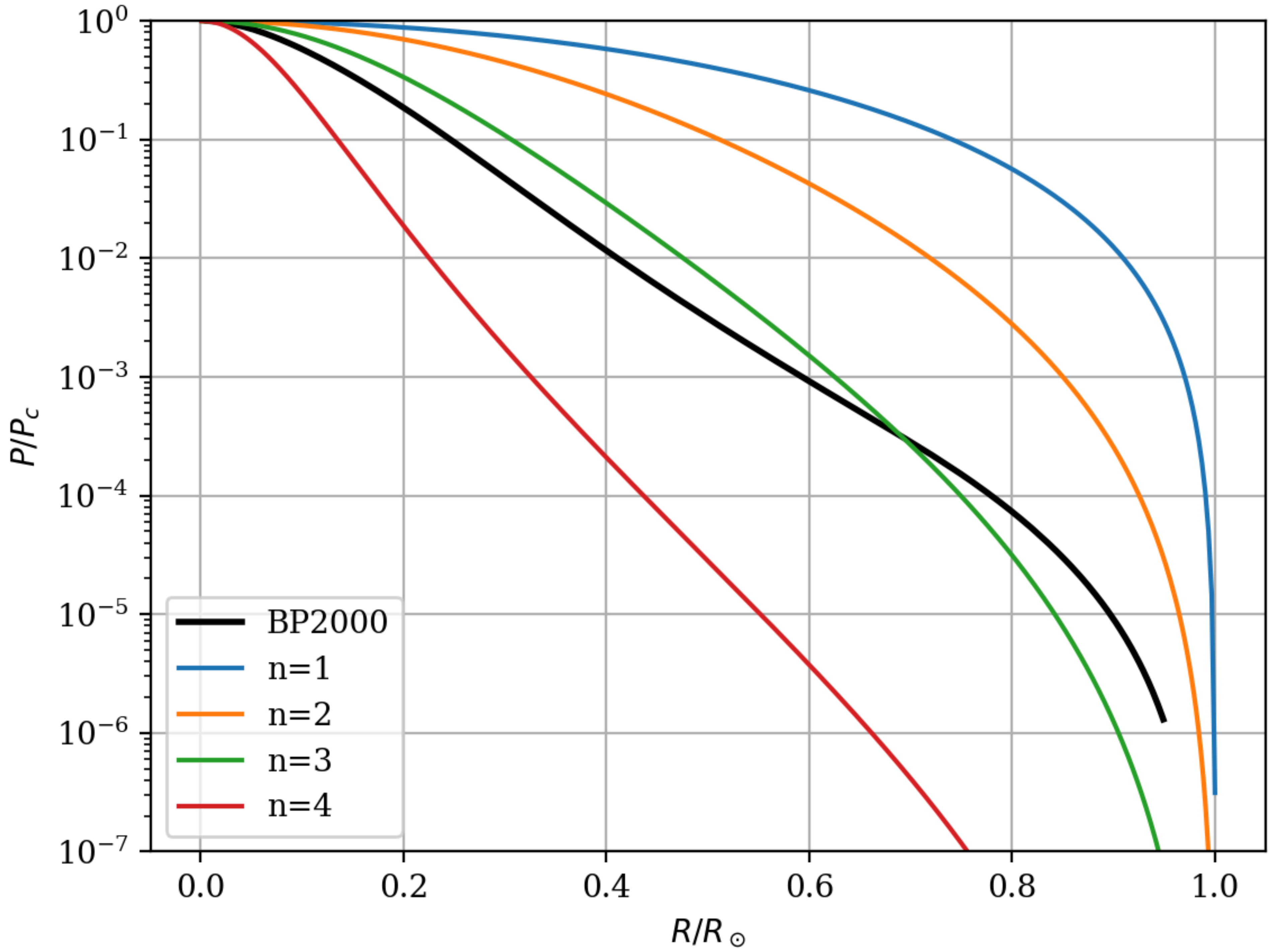


- **Higher n: density more heavily towards the center!**



Lane-Emden Equation

- Q: Why are we using these again?
  - Computers creating realistic stellar interiors is STILL hard
  - Polytropes are a good first assumption for the internal structure of unknown bodies (e.g. exoplanets)



- They’re in Homework 4!